Method of characteristics

Before we begin to understand the solutions for the exercises in this problem sheet, we first briefly describe the method of characteristics for solving non-linear first-order partial differential equations. To keep the discussion concise, we forgo the rigorous arguments, although, the arguments presented can be made rigorous (for details see [1, 2]).

Let us consider the following PDE

$$\frac{\partial u}{\partial t} + b(x,t,u)\,\frac{\partial u}{\partial x} = c(x,t,u)\,, \quad x \in \mathbb{R}, t \in \mathbb{R}^+ \tag{1}$$

with an initial condition

$$u(x,0) = u_0(x)\ . \tag{2}$$

While this is not the most general setup the method can treat, it is general enough for our purpose. The main idea of the method of characteristics is to find curves in the $x-t$ plane, along which the PDE reduces to an appropriate system of first-order ODEs. These curves are also known as *characteristics.*

For a fixed x-intercept ξ, consider the following ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = b(x,t,u) \qquad\qquad x(0) = \xi. \tag{3}$$

Under sufficient regularity conditions, one can find a unique solution of (3) as $\hat{x}(\xi,t) = x(t;\xi)$ (parameterized by the initial x-intercept). Thus, along the curve $\{\hat{x}(\xi,t),t\}$, the PDE (1) reduces to the following ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}(\xi,t) = c(\hat{x},t,\hat{u}) \qquad\qquad \hat{u}(\xi,0) = u_0(\xi)\,, \tag{4}$$

where $\hat{u}(\xi,t) = u(\hat{x}(\xi,t),t) = u(x(t;\xi),t)$. Again, under sufficient regularity conditions we can find a unique solution for (4).

Finally, we need to express the solution in terms of the (x,t). In order to do this, we need to solve for ξ from the equation $x = \hat{x}(\xi,t)$, i.e., we need to find a smooth function $\hat{\xi} = \hat{\xi}(x,t)$ such that

$$x = \hat{x}\Big(\hat{\xi}(x,t)\,,t\Big)\,. \tag{5}$$

The existence of a function $\hat{\xi}(x,t)$ satisfying (5) is ensured if $\hat{x}_\xi \neq 0$, due to the inverse function theorem. Thus, the solution to (1),(2) is given by

$$u(x,t) = \hat{u}\Big(\hat{\xi}(x,t),t\Big)\,. \tag{6}$$

References

[1] Partial Differential Equations, by Lawrence. C. Evans. *Chapter 3, Section 2.*

[2] Partial Differential Equations, by Fritz John.